Fractal-Based Point Processes

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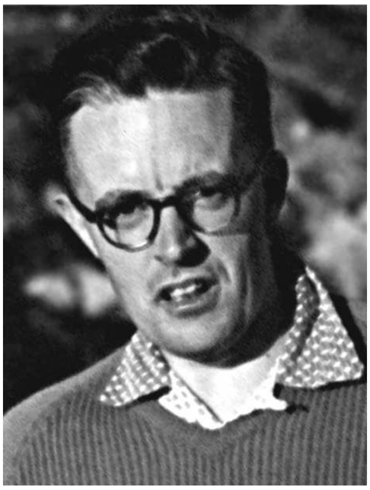
Point Processes: Examples

The French physiologist **Louis Lapicque (1866–1952)** conceived the integrate-and-reset point process; it successfully describes the generation of action potentials by a broad variety of neurons and continues to enjoy wide use today.



Sir David R. Cox (born 1924), a British statistician, studied the superposition of periodic series of events and, as part of his work in the textile industry in the 1940s, conceived the doubly stochastic Poisson process.

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Having set forth a collection of measures useful for examining point processes in Chapter 3, we now consider a number of examples. Although the examples we provide do not exhibit fractal behavior *per se*, they do play an important role in the construction of fractal and fractal-rate point processes, as we will see subsequently.

We consider the following processes, in turn[1]: homogeneous Poisson point processes (Sec. 4.1), renewal point processes (Sec. 4.2), doubly stochastic Poisson point processes (Sec. 4.3), integrate-and-reset point processes (Sec. 4.4), cascaded point processes (Sec. 4.5), branching point processes (Sec. 4.6), and Lévy dusts (Sec. 4.7). A broad range of other point processes also finds use in characterizing many diverse phenomena (see, for example, Bartlett, 1955; Parzen, 1962; Cox & Lewis, 1966; Feller, 1971; Lewis, 1972; Srinivasan, 1974; Saleh, 1978; Cox & Isham, 1980; Snyder & Miller, 1991).

4.1 HOMOGENEOUS POISSON POINT PROCESS

We begin with the one-dimensional **homogeneous Poisson process**, which arises under a broad variety of circumstances (Parzen, 1962; Cox, 1962; Haight, 1967; Cox & Isham, 1980). As indicated in Sec. 2.5.2, the definition of this process consists of two parts. First, for some fixed, constant mean rate μ, we have

$$\mathrm{E}[N(t+s) - N(s)] = \mu t, \tag{4.1}$$

independent of the times s and t. Second, events in nonoverlapping segments do not depend on one another; formally the two differences

$$N(t_2) - N(t_1) \qquad \text{and} \qquad N(t_4) - N(t_3) \tag{4.2}$$

remain independent for any t_1, t_2, t_3, t_4, satisfying $t_1 < t_2 \leq t_3 < t_4$.

Conny Palm[2] (1943) was the first to point out that this point process is "without aftereffects." As a consequence of its "zero-memory" behavior, both the intervals

[1] We consider one-dimensional constructs, although most of these processes have **multidimensional point process** counterparts (see, for example, Fisher, 1972; Cox & Isham, 1980, Chapter 6).
[2] A photograph of Palm is placed at the beginning of Chapter 13.

$\{\tau_k\}$ *and* the counts $\{Z_k\}$ form sequences of independent, identically distributed random variables. Because of its simplicity, the homogeneous Poisson process serves as a benchmark against which other stochastic point processes are often compared. It plays the role that the white Gaussian process enjoys in the realm of continuous-time stochastic processes.

A number of other properties follow from the definition provided above (Cox & Isham, 1980). The times between events follow a decaying exponential probability density function[3]

$$p_\tau(t) = \begin{cases} \mu \exp(-\mu t) & \text{for } t > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

with associated moments

$$\mathrm{E}\left[\tau^k\right] = k!/\mu^k. \tag{4.4}$$

In particular, $\mathrm{E}[\tau] = \mu^{-1}$ and $\mathrm{Var}[\tau] = \mu^{-2}$ so that $C_\tau = \sqrt{\mathrm{Var}[\tau]}/\mathrm{E}[\tau] = 1$; this simple result supports the use of the homogeneous Poisson process as a benchmark.

The interval-based autocorrelation and spectrum assume simple forms as a result of the independence of the intervals:

$$R_\tau(k) = \begin{cases} 2\mu^{-2}, & k = 0 \\ \mu^{-2}, & k \neq 0 \end{cases} \tag{4.5}$$

$$S_\tau(\mathsf{f}) = \mu^{-2}\,\delta(\mathsf{f}) + \mu^{-2}, \tag{4.6}$$

while rescaled range and detrended fluctuation analyses follow the forms given in Secs. 3.3.5 and 3.3.6 for independent intervals.

The number of counts over a fixed time follows the distribution set forth by Poisson[4] in 1837:

$$\Pr\{N(t+s) - N(s) = n\} = \Pr\{Z(t) = n\} = (\mu t)^n \, \exp(-\mu t)/n!. \tag{4.7}$$

Interestingly, this now-famous distribution aroused little interest until 1898 when von Bortkiewicz wrote a monograph providing a whole host of examples[5] for which the Poisson distribution was applicable (see Quine & Seneta, 1987, for a discussion).

The factorial moments of this distribution are

$$\mathrm{E}\left\{ \frac{[Z(t)]!}{[Z(t)-k]!} \right\} = (\mu t)^k. \tag{4.8}$$

[3] Despite the formal distinction between the density and distribution functions, we often refer to both simply as "distributions."

[4] The term "Poisson" conventionally denotes both the point process itself and the distribution of the number of counts in the process. Since the same term refers to two quite different mathematical constructs, we generally use the full terms to avoid confusion: "homogeneous Poisson process" (or "homogeneous Poisson" for short) and "Poisson distribution," respectively.

[5] The most celebrated among these, perhaps, is von Bortkiewicz's (1898) analysis of the number of deaths from horse kicks in the Prussian army.

In particular, $\mathrm{E}[Z(t)] = \mathrm{Var}[Z(t)] = \mu t$. Further, we have

$$\begin{aligned}
F(T) = A(T) &= 1 && \text{a)} \\
R_Z(k,T) &= \begin{cases} \mu T + \mu^2 T^2, & k = 0 \\ \mu^2 T^2, & k \neq 0 \end{cases} && \text{b)} \\
S_N(f) &= \mu^2 \, \delta(f) + \mu && \text{c)} \\
G(t) &= \mu \, \delta(t) + \mu^2 && \text{d)} \\
\mathrm{Var}\left[C_{\psi,N}(a,b)\right] &= \mu \int \psi^2(x) \, dx && \text{e)} \\
D_q &= 0. && \text{f)}
\end{aligned} \tag{4.9}$$

The quantity μ that appears in Eqs. (4.1) and (4.3)–(4.9) takes the same value in each equation.

The homogeneous Poisson process successfully models a whole host of phenomena over short times, including radioactive decay (Sec. 2.5.4), the commencement of telephone conversations at large exchanges (Sec. 13.1), and the times at which falling raindrops hit the ground. These phenomena have in common the combination of events from many independent sources, so that over a short time no single source contributes significantly to the total set of events.

This broad range of applications of the homogeneous Poisson process highlights the convergence property of superpositions of point processes, which we now examine. Formally, we begin with a collection of independent counting processes $\{N_{1,k}(t)\}$, each with a mean rate $\mu_{1,k}$. Consider the sum of the first M of these processes

$$N_{2,M}(t) \equiv \sum_{k=1}^{M} N_{1,k}(t), \tag{4.10}$$

which has a total rate

$$\mu_{2,M} = \sum_{k=1}^{M} \mu_{1,k}. \tag{4.11}$$

Now scale the time axis by a factor of $\mu_0/\mu_{2,M}$, where μ_0 is any fixed constant rate. This yields

$$N_{3,M}(t) \equiv N_{2,M}(t\mu_0/\mu_{2,M}) = \sum_{k=1}^{M} N_{1,k}(t\mu_0/\mu_{2,M}); \tag{4.12}$$

the process $N_{3,M}(t)$ has a rate μ_0 for all M.

In the limit $M \to \infty$, assuming $\mu_{2,M} \to \infty$, the superposition $N_{3,M}(t)$ approaches a homogeneous Poisson process with rate μ_0 (Palm, 1943; Cox & Smith, 1953, 1954; Khinchin, 1955; Grigelionis, 1963; Franken, 1963, 1964; Çinlar, 1972; Franken, König, Arndt & Schmidt, 1981). We can readily understand this from an intuitive point of view: for large M each of the point processes $N_{1,k}(t)$ contributes few events to $N_{3,M}(t)$ over any finite time interval $[0, t)$, and in the limit $M \to \infty$ no single process contributes more than a single event. The events are therefore completely independent of each other, whereupon the homogeneous Poisson process results.

In addition to the zero-memory and superposition approaches considered above, many other routes also lead to the homogeneous Poisson process. One example is sparse random selection from an arbitrary point process (Cox & Isham, 1980), a topic considered further in Sec. 11.2.3.

4.2 RENEWAL POINT PROCESSES

The independence property of the homogeneous Poisson process tells us that the set of intervals between adjacent events $\{\tau_n\}$ are independent and identically distributed. This provides an alternate definition of this process: an independent and identically distributed set $\{\tau_n\}$ with a probability density function specified by Eq. (4.3).

A ready generalization of the homogeneous Poisson process lies in choosing an arbitrary interevent-interval probability density function while retaining its independent and identically distributed features. The result is a **renewal point process**; the name derives from the fact that the process begins anew (undergoes a renewal) at the occurrence of each event. Renewal processes are often used to describe the behavior of parts such as light bulbs since the failure of one part results in its replacement with a replica chosen at random with an identical *a priori* lifetime distribution (Lotka, 1939; Feller, 1941; Doob, 1948; Smith, 1958; Takács, 1960; Parzen, 1962; Cox, 1962; Cox & Isham, 1980). The origins of renewal theory lie in the life tables of the citizens of London and Breslau published in the late 1600s (see Daley & Vere-Jones, 1988, Chapter 1).

As with other point processes treated in this book, we generally consider stationary versions of renewal point processes, in the sense of Eq. (3.2). For renewal point processes only, the term "equilibrium" means stationary, whereas the term "pure" denotes a renewal point processes that begins with an event.

For any renewal point process the sequence of intervals $\{\tau_k\}$ exhibits independence by construction; this leads to simple forms for second-order interval-based statistics. However, such simplicity does not extend to other measures, such as the coincidence rate, the spectrum of the point process, or statistics derived from the sequence of counts, $\{Z_k(T)\}$. Nevertheless, explicit expressions exist that quantify the characteristics of the renewal point process in terms of the interevent-interval probability density function $p_\tau(t)$.

We begin the study of stationary renewal point processes with construction of the coincidence rate. The density function $p_\tau(t)$ itself describes the probability of an event occurring at a time t given an event at the origin, with no intervening events. To obtain the probability of an event occurring at a time t given an event at the origin, with exactly one intervening event, we simply add the two (independent) random variables. The corresponding probability density is then simply the convolution of

$p_\tau(t)$ with itself

$$p_\tau^{\star 2}(t) = p_\tau(t) \star p_\tau(t) = \int_0^t p_\tau(t-s)\, p_\tau(s)\, ds, \tag{4.13}$$

where $\star$ denotes the convolution operation and $p_\tau^{\star 2}(t)$ represents $p_\tau(t)$ convolved with itself.

Continuing in this same way yields

$$p_\tau^{\star n}(t) = p_\tau^{\star(n-1)}(t) \star p_\tau(t) = \int_0^t p_\tau^{\star(n-1)}(t-s)\, p_\tau(s)\, ds \tag{4.14}$$

for precisely $n-1$ intervening events, where $p_\tau^{\star n}(t)$ represents $p_\tau(t)$ convolved with itself n times, and we employ the notational convenience $p_\tau^{\star 0}(t) = \delta(t)$, the Dirac delta function. Summing over all possible numbers of intervening events, normalizing by the conditional probability of an event at $t = 0$, and admitting negative values of t yields the coincidence rate for a renewal point process (Feller, 1971; Lowen & Teich, 1993d),

$$G(t) = \mathrm{E}[\mu] \sum_{n=0}^{\infty} p_\tau^{\star n}(|t|). \tag{4.15}$$

We can obtain the spectrum of a renewal point process (Lukes, 1961) via the Fourier transform of Eq. (4.15):

$$\begin{aligned}
S_N(f) &\equiv \int_{-\infty}^{\infty} e^{-i2\pi ft}\, \mathrm{E}[\mu] \sum_{n=0}^{\infty} p^{\star n}(|t|)\, dt \\
&= \mathrm{E}[\mu] + 2\mathrm{E}[\mu]\, \mathrm{Re}\left\{ \int_0^\infty e^{-i2\pi ft} \sum_{n=1}^{\infty} p^{\star n}(t)\, dt \right\} \\
&= \mathrm{E}[\mu] + 2\mathrm{E}[\mu]\, \mathrm{Re}\left\{ \sum_{n=1}^{\infty} \int_0^\infty e^{-i2\pi ft} p^{\star n}(t)\, dt \right\} \\
&= \mathrm{E}[\mu] + \mathrm{E}^2[\mu]\, \delta(f) + 2\mathrm{E}[\mu]\, \mathrm{Re}\left\{ \sum_{n=1}^{\infty} \phi_\tau^n(2\pi f) \right\} \\
&= \mathrm{E}^2[\mu]\, \delta(f) + \mathrm{E}[\mu]\, \mathrm{Re}\left\{ \frac{1-\phi_\tau(2\pi f)}{1-\phi_\tau(2\pi f)} \right\} + 2\mathrm{E}[\mu]\, \mathrm{Re}\left\{ \frac{\phi_\tau(2\pi f)}{1-\phi_\tau(2\pi f)} \right\} \\
&= \mathrm{E}^2[\mu]\, \delta(f) + \mathrm{E}[\mu]\, \mathrm{Re}\left\{ \frac{1+\phi_\tau(2\pi f)}{1-\phi_\tau(2\pi f)} \right\},
\end{aligned} \tag{4.16}$$

where the characteristic function of the interevent intervals $\phi_\tau(\omega)$ is defined in Eq. (3.6), $\mathrm{Re}[z]$ represents the real part of the complex expression z, and the delta function in Eq. (4.16) derives from the constant $\mathrm{E}^2[\mu]$ term in the coincidence rate. In the low-frequency limit (Lowen, 1992), this reduces to (see Prob. 4.5)

$$\lim_{f \to 0} S_N(f) = \mathrm{E}^3[\mu]\, \mathrm{Var}[\tau]. \tag{4.17}$$

Substituting Eq. (4.17) into Eq. (3.64) yields

$$\lim_{T\to\infty} F(T) = \lim_{T\to\infty} A(T) = \mathrm{E}^2[\mu]\,\mathrm{Var}[\tau] = C_\tau^2. \tag{4.18}$$

Feller (1968, Sec. XIII.6, pp. 320–322) obtained this important result for renewal processes by other means.

Making use of Fourier and z transforms (Lowen, 1992) yields an expression for a type of factorial moment for renewal processes (see Sec. A.2.1)

$$\begin{aligned}\mathrm{E}\left\{\frac{[Z(T)+k-1]!}{[Z(T)-1]!}\right\} &= \mathrm{E}\big\{Z(T)\,[Z(T)+1]\cdots[Z(T)+k-1]\,\big\}\\ &= \mathrm{E}^{2-k}[\mu]\,k!\int_{0-}^{T}(T-t)\,G^{\star(k-1)}(t)\,dt. \end{aligned}\tag{4.19}$$

In particular, substituting $k=1$ into Eq. (4.19) yields

$$\mathrm{E}[Z(T)] = \mathrm{E}[\mu]\,T, \tag{4.20}$$

a canonical result for all point processes, while $k=2$ and some algebra provides (see Prob. 4.6)

$$\mathrm{Var}[Z(T)] = \int_{-T}^{T}(T-|t|)\left\{G(t)-\mathrm{E}^2[\mu]\right\}\,dt, \tag{4.21}$$

recalling Eq. (3.52), another result general to all point processes. However, larger values of k in Eq. (4.19) apply only to renewal point processes, and not to general point processes.

Renewal point processes with interevent intervals that have power-law distributions, as considered in Chapter 7, are known as **fractal renewal point processes**. Fractal-based point processes can also be derived from collections of **alternating fractal renewal processes**, as considered in Chapter 8.

4.3 DOUBLY STOCHASTIC POISSON POINT PROCESSES

Another generalization of the homogeneous Poisson process emerges when the rate μ is modulated. The **doubly stochastic Poisson process** results from choosing $\mu(t)$ to be a positive-valued continuous-time stochastic rate process rather than a fixed constant. The resultant process is thus *doubly* random: an (unobserved) source of randomness arises from the fluctuations in the stochastic rate $\mu(t)$ while another source arises from the intrinsic Poisson event-generation fluctuations, given the rate $\mu(t)$.[6]

[6] The designation "mixed Poisson process," initially used by Bartlett (1955), signifies that the rate is a random variable (fixed in time) rather than a random process (varying in time). On occasion the term "compound Poisson process" appears in place of "doubly stochastic Poisson process" but this terminology is generally reserved for describing cascaded Poisson processes (see Sec. 4.5).

This stochastic point process was conceived by David Cox (1955) to describe the sequence of stops of a loom in a textile mill.[7] This sequence would ordinarily be expected to form a Poisson process with a fixed rate of stoppage. However, random variations of the quality of the material provided to the loom lead to fluctuations in the stoppage rate. Since its development, the doubly stochastic Poisson process has found wide application in a broad variety of fields (see, for example, Bartlett, 1963; Cox & Lewis, 1966; Lewis, 1972; Grandell, 1976; Saleh, 1978; Cox & Isham, 1980; Saleh & Teich, 1982; Saleh, Stoler & Teich, 1983; Teich & Saleh, 1988; Snyder & Miller, 1991; Teich & Saleh, 2000).

Formally, we have

$$\lim_{\epsilon\to 0} \epsilon^{-1} \Pr\left\{N(t+\epsilon) - N(t) > 0 \,\middle|\, \mu(t)\right\} = \mu(t). \tag{4.22}$$

Figure 4.1 presents a realization of this point process. The fluctuations exhibited in the rate (a) appear in random form in the ensuing point-process events displayed in (b). Two statistics follow immediately from Eq. (4.22). Taking expectations of both sides yields $\mathrm{E}[dN(t)/dt] = \mathrm{E}[\mu]$, and integration leads to

$$\mathrm{E}[N(t)] = \mathrm{E}[\mu]t. \tag{4.23}$$

Similarly, employing Eq. (4.22) at two different times gives rise to the coincidence rate

$$\begin{aligned} \mathrm{E}\left[\frac{dN(s)}{ds}\,\frac{dN(s+t)}{ds}\right] &= \mathrm{E}[\mu(s)\,\mu(s+t)] \\ G(t) &= R_\mu(t) + \mathrm{E}[\mu]\,\delta(t), \end{aligned} \tag{4.24}$$

where $R_\mu(t)$ denotes the autocorrelation of $\mu(t)$. Taking the Fourier transform leads directly to

$$S_N(f) = S_\mu(f) + \mathrm{E}[\mu], \tag{4.25}$$

where $S_\mu(f)$ represents the spectrum of the rate $\mu(t)$.

Other statistics of this point process derive from those of the rate process $\mu(t)$. In parallel with Eqs. (4.7) and (4.8), we have (Saleh, 1978)

$$\Pr\{Z(t) = n\} = \mathrm{E}\{\Lambda(t)\ \exp[-\Lambda(t)]\}/n! \tag{4.26}$$

and

$$\mathrm{E}\left\{\frac{[Z(t)]!}{[Z(t)-k]!}\right\} = \mathrm{E}\left[\Lambda^k(t)\right], \tag{4.27}$$

where we have defined the integrated rate

$$\Lambda(t) = \int_0^t \mu(s)\,ds. \tag{4.28}$$

[7] The process is also known as a **Cox process**. The appellation “doubly stochastic Poisson process,” often abbreviated DSPP, was provided by Bartlett (1963).

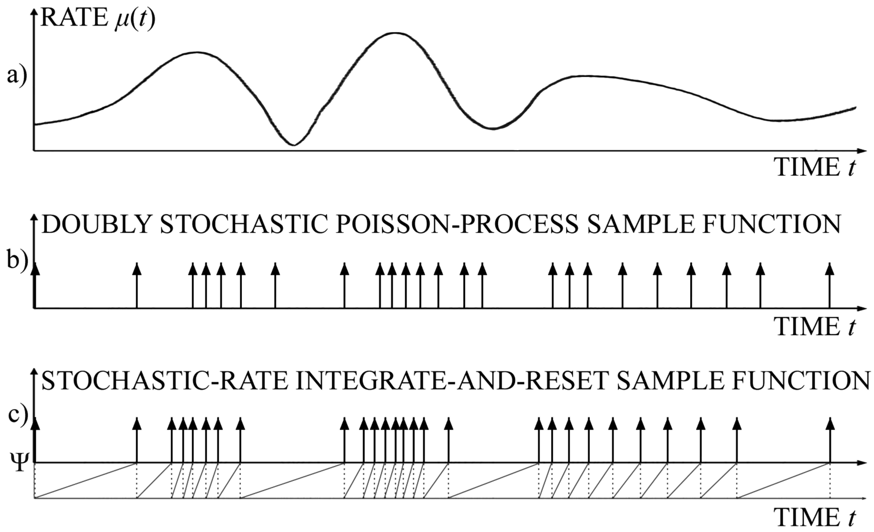

Fig. 4.1 a) Sample function of a stochastic rate $\mu(t)$. This realization serves as the rate for a Poisson point process and an integrate-and-reset point process, as considered in turn below. b) Sample function of the resulting doubly stochastic Poisson process. Events tend to occur more often when the rate $\mu(t)$ assumes larger values, although the randomness introduced by the Poisson process renders this association probabilistic. c) The integral of the rate increases until it reaches a threshold $\Psi = 1$, whereupon it resets to zero and the integration begins anew (gray). The reset times form a point process known as the stochastic-rate integrate-and-reset point process (black).

In particular,

$$\Pr\{Z(t) = 0\} = \mathrm{E}\left\{\exp\left[-\int_0^t \mu(s)\, ds\right]\right\}. \tag{4.29}$$

Equation (4.29), together with Eq. (3.30), yields the interval density

$$\begin{aligned} p_\tau(t) &= \mathrm{E}[\tau]\, \frac{d^2}{dt^2}\, \mathrm{E}\left\{\exp\left[-\int_0^t \mu(s)\, ds\right]\right\} \\ &= \frac{1}{\mathrm{E}[\mu]}\, \mathrm{E}\left\{\left[\mu^2(t) - \frac{d\mu(t)}{dt}\right] \exp\left[-\int_0^t \mu(s)\, ds\right]\right\}. \end{aligned} \tag{4.30}$$

When the rate process $\mu(t)$ exhibits fluctuations over frequency ranges that are significantly lower than the mean rate $\mathrm{E}[\mu]$, the interval density $p_\tau(t)$ takes a simpler form, and a straightforward expression for the moments of τ emerges (Saleh, 1978):

$$\mathrm{E}[\tau^n] \approx n!\, \mathrm{E}[\mu^{1-n}]/\mathrm{E}[\mu] \tag{4.31}$$

$$p_\tau(t) \approx \mathrm{E}\big[\mu^2 \exp(-\mu t)\big]/\mathrm{E}[\mu]. \tag{4.32}$$

A particularly simple result obtains when the coefficient of variation C_μ of this rate process $\mu(t)$ becomes small in comparison with unity. As $C_\mu \to 0$, the stochastic rate process $\mu(t)$ approaches a constant, deterministic value. The expectation operators in Eq. (4.32) then become superfluous, whereupon it simplifies to Eq. (4.3). Moreover, since the rate remains constant, the resulting point process $dN(t)$ becomes the homogeneous Poisson process. For rate processes $\mu(t)$ with a small, but nonzero, coefficient of variation ($0 < C_\mu \ll 1$), the interval density approaches the exponential form

$$p_\tau(t) \approx \begin{cases} \mathrm{E}[\mu] \exp\big(-\mathrm{E}[\mu]\, t\big) & \text{for } t > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.33}$$

However, in this case the variation of $\mu(t)$ imparts memory to $dN(t)$, so that the *ordering* of the intervals differs from that of the homogeneous Poisson process. The resulting point process is therefore nonrenewal.

Although most doubly stochastic Poisson processes are, in fact, nonrenewal in nature, the classes of renewal point processes and doubly stochastic Poisson processes do intersect. In particular, for any distribution $P(t)$, and any positive constant time t_c, the quantity

$$\phi_\tau(\omega) = \left[1 + i\omega t_c + \int_0^\infty \left(1 - e^{-i\omega t}\right)\, dP(t)\right]^{-1} \tag{4.34}$$

defines the characteristic function of an interevent-interval distribution $P_\tau(t)$. A renewal point process constructed with this interevent-interval distribution will also be a doubly stochastic Poisson point processes (Grandell, 1976). Thus, two quite different models generate identical behavior. Conversely, it is impossible to distinguish between the two models in this case, even with a full description of the process itself (the probabilities of all possible outcomes over all time).

This highlights the paucity of information generally available in point processes in comparison with continuous functions of time. The set of event times contained within a finite interval, for example, completely describes the point process during that interval. With probability one, this set has a finite dimension. In contrast, the set of continuous functions over any finite interval forms an infinite-dimensional set. As we shall see in Chapter 12, this relative sparseness makes identification of an underlying model quite difficult in practice.

Point processes whose rate functions comprise stochastic processes are examined in Chapter 10. In particular, if the rate function is **shot noise** (see Chapter 9), the associated doubly stochastic Poisson process is known as a **shot-noise-driven doubly stochastic Poisson process** (Saleh & Teich, 1982). If the rate function is **fractal shot noise** (see Chapter 9), the process is called a **fractal-shot-noise-driven doubly stochastic Poisson process** (Lowen & Teich, 1991). Point processes in this class find use in characterizing a multitude of phenomena in the physical and biological sciences.

4.4 INTEGRATE-AND-RESET POINT PROCESSES

The **integrate-and-reset**, or **integrate-and-fire**, model was introduced nearly a century ago by Lapicque (1907, 1926). This simple nonlinear construct provides a direct route for transforming a rate function into a point process.

Integrate-and-reset point processes play an important role in modeling biophysical phenomena, particularly neural spike trains, since the paradigm offers not only a suitable mathematical model, but also a plausible physiological model for the underlying behavior (Eccles, 1957; Holden, 1976; Tuckwell, 1988; Koch, 1999). These processes are closely related to **oversampled sigma-delta modulators** in the domain of signal processing, where they are used for analog-to-digital conversion (Norsworthy, Schreier & Temes, 1996).

Like the doubly stochastic Poisson process, the integrate-and-reset point process depends on a stochastic rate $\mu(t)$. However, in this case the sole source of randomness manifested in the generated point process arises from fluctuations associated with the rate $\mu(t)$; the integrate-and-reset algorithm introduces no additional randomness of its own.

The algorithm generates an event each time the integral of the rate $\mu(t)$ reaches a value of unity. It then resets the integrated value to zero whereupon the process begins anew. Formally, we have

$$t_{k+1} = \inf_{u>t_k} \left\{ u : \int_{t_k}^{u} \mu(s)\, ds = 1 \right\}, \tag{4.35}$$

where $\{t_k\}$ again represents the set of times at which the events occur (rather than the times between events). A realization of this point process is presented in Fig. 4.1, where the resulting point-process events (c) are seen to faithfully follow the fluctuations of the rate (a) to within the resolution of the point process.

The absence of additional randomness leads to trivial results for simple forms of the rate $\mu(t)$; in particular, for $\mu(t)$ a fixed constant value μ, the resulting point process comprises a perfectly periodic train of events spaced from each other by $1/\mu$. The faithfulness of the transformation also leads to a close correspondence between the second-order measures of $\mu(t)$ and those of $dN(t)$, particularly for large times (low frequencies). For example, when $f \ll \mathrm{E}[\mu]$, we obtain

$$S_N(f) \approx S_\mu(f), \tag{4.36}$$

in contrast to the corresponding result for the doubly stochastic Poisson process, Eq. (4.25), which contains an additional term, $\mathrm{E}[\mu]$, associated with the intrinsic randomness of the underlying Poisson process. Exact expressions prove difficult to obtain for the integrate-and-reset process, however, by virtue of its inherent nonlinearity.

Again, simple forms emerge for the interevent-interval statistics when the rate process $\mu(t)$ exhibits fluctuations significantly slower than the mean rate of events $\mathrm{E}[\mu]$. In the spirit of Eq. (4.32), we begin with the interevent interval probability density function for a fixed rate process μ, namely $\delta(t-1/\mu)$, and include appropriate weighting and normalization factors,

$$\begin{aligned} p_\tau(t) &\approx \mathrm{E}[\mu\,\delta(t-1/\mu)]\,/\mathrm{E}[\mu] \\ &= \int_0^\infty y\;\delta(t-1/y)\,p_\mu(y)\,dy/\mathrm{E}[\mu] \\ &= \mathrm{E}[\mu]^{-1}\int_0^\infty y^2\,t^{-1}\,\delta(y-1/t)\,p_\mu(y)\,dy \\ &= \mathrm{E}[\mu]^{-1}\,t^{-3}\,p_\mu(1/t), \end{aligned} \tag{4.37}$$

where we have made use of a particular property of the Dirac delta function, namely $\delta(ax) = a^{-1}\delta(x)$. We note that under these conditions the output point process is generally not renewal because the reset mechanism does not erase the history of the input process, which is preserved through the fluctuations in $\mu(t)$.

We can directly obtain the moments of the interval density from the moments of the rate:

$$\begin{aligned} \mathrm{E}[\tau^n] &\approx \mathrm{E}[\mu]^{-1}\int_0^\infty t^{n-3}\,p_\mu(1/t)\,dt \\ &= \mathrm{E}[\mu]^{-1}\int_0^\infty y^{1-n}\,p_\mu(y)\,dy \\ &= \mathrm{E}[\mu^{1-n}]/\mathrm{E}[\mu]. \end{aligned} \tag{4.38}$$

In particular, Eq. (4.38) yields the interval standard deviation, and thence the coefficient of variation, for an ideal integrate-and-reset point process driven by an arbitrary stochastic rate, provided that the fluctuations of the rate process are sufficiently slow (see Prob. 4.9).

For simplicity, we chose the threshold to be unity ($\Psi = 1$) in Eq. (4.35). Any positive value would have sufficed without changing the nature of the process, serving only to divide the rate by the new threshold.

On the other hand, if the threshold varies in time, the character of the resulting integrate-and-reset point process $dN(t)$ changes substantially. The rationale for considering models of this form stems from early neurophysiological experiments in which it was demonstrated that a sequence of identical brief electric currents applied to a neuron near threshold elicited axonal action-potential responses only in a fraction of the trials, in random fashion (Blair & Erlanger, 1932, 1933). Behavior of this kind has been ascribed to fluctuations in threshold (Pecher, 1939; Verveen, 1960; Holden, 1976), also referred to as “fluctuations in excitability.”

A sinusoidally varying threshold, for example, imparts its fluctuations to $dN(t)$, albeit in a nonlinear fashion. The complex interplay between the rate and the threshold, when both vary, forms a rich field of study. We consider two examples with variable thresholds in this text. The first comprises a threshold that remains fixed during each integration, but assumes an independent, unit-mean exponential random value for every event. The resulting point process is then indistinguishable from a doubly stochastic Poisson process with the same rate function, by virtue of the exponential interevent-interval density function for the homogeneous Poisson process.

The other example, which appears in Sec. 6.6, gives rise to a fractal-based point process from regular Brownian motion; it has found use in modeling the fractal-rate fluctuations of neural spike trains.

Finally, we mention a generalization of this process, known as the **leaky integrate-and-reset process** (Lapicque, 1907; Holden, 1976; Tuckwell, 1988; Park & Gray, 1992). In this case, an internal state variable x increases at a rate proportional to the instantaneous rate $\mu(t)$, while simultaneously decaying to zero with a time constant t_c. When x reaches unity, an output event is generated, x is reset to zero, and the cycle begins anew. The state equation is written as

$$dx/dt = \mu(t) - x/t_c, \tag{4.39}$$

or, equivalently,

$$x(t) = \exp(-t/t_c) \int_0^t \mu(s)\, \exp(s/t_c)\, ds. \tag{4.40}$$

In the limit $t_c \to \infty$ we recover the behavior of Eq. (4.35). The added flexibility provided by the formalism of Eqs.(4.39) and (4.40) proves useful in some applications; however, the added mathematical complexity does not warrant our considering it further here.

4.5 CASCADED POINT PROCESSES

Cascaded point processes, also known as **cluster point processes** (Neyman & Scott, 1958, 1972; Saleh & Teich, 1983) and **compound point processes** (Feller, 1968, Chapter 12), arise when each event of a point process forms the nucleus for a sequence of secondary point-process segments.

In the most general formulation, each event t_k of the primary point process $dN_1(t)$ initiates a secondary point process $dN_{2,k}(t)$, which terminates after a random number $M_k = N_{2,k}(\infty)$ of events. All secondary points, taken together as indistinguishable events, form the output point process $dN_3(t)$. Figure 4.2 illustrates this construct. The primary events can be excluded or included with the secondary processes.

The statistics for a cascaded point process necessarily depend on the details of the primary and secondary processes that comprise it. General closed-form expressions do not exist, with a single exception: the mean rate for a stationary process. Consider a primary point process $dN_1(t)$ that generates events at a mean rate $\mathrm{E}[\mu_1]$. Each such primary event initiates a secondary process with a mean number $\mathrm{E}[M_k]$ events. For the mean rate $\mathrm{E}[\mu_3]$ of the cascaded point process $dN_3(t)$ itself, we arrive at

$$\mathrm{E}[\mu_3] = \mathrm{E}[\mu_1]\, \mathrm{E}[M_k]\,. \tag{4.41}$$

Within the class of cascaded point processes, two forms have been studied extensively. For both, the homogeneous Poisson process forms the primary process and the clusters are independent of each other. In the **Neyman–Scott cluster process**

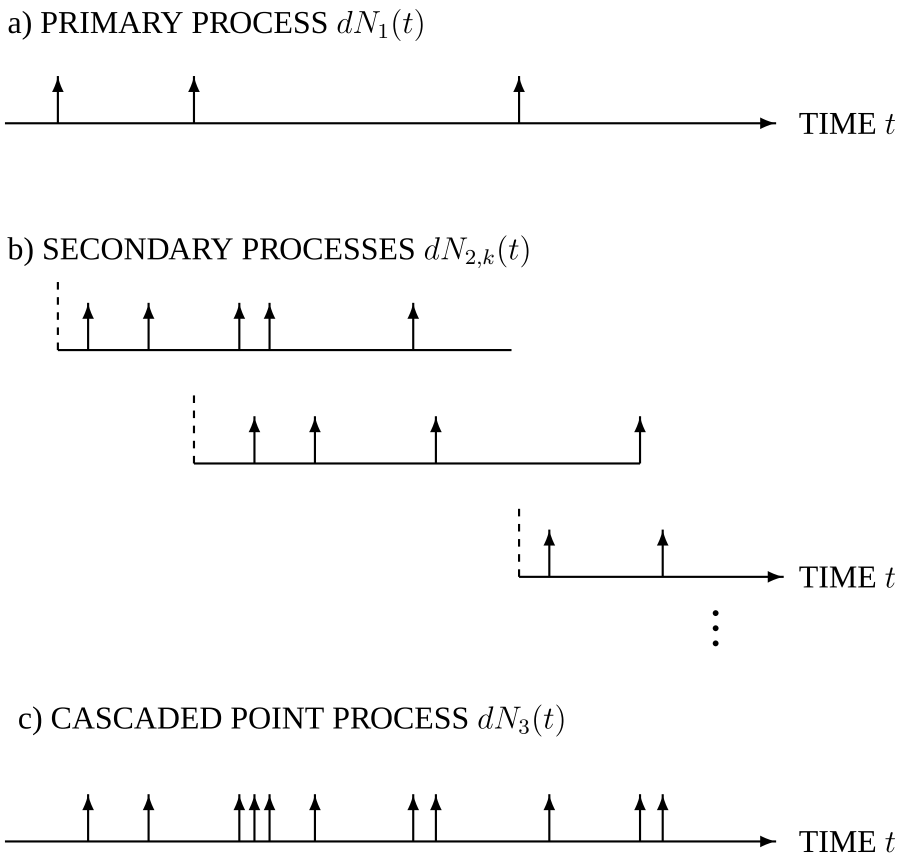

Fig. 4.2 Generation of a cascaded point process. Each event of a primary point process $dN_1(t)$ (a) initiates a secondary point process $dN_{2,k}(t)$ (b) that terminates after a random number of events. All secondary points, taken together as indistinguishable events, form the output, which is a cascaded point process $dN_3(t)$ (c). The primary events may be excluded or included in the output. Cascaded point processes are also known as compound processes or cluster processes.

(Neyman & Scott, 1958; Bartlett, 1964; Vere-Jones, 1970; Neyman & Scott, 1972; Saleh & Teich, 1982, 1983; Daley & Vere-Jones, 1988), the times between each secondary event and its corresponding primary event are independent and identically distributed. In the **Bartlett–Lewis cascaded process**[8] (Bartlett, 1963; Lewis, 1964, 1967), each primary event initiates a segment of a renewal point process, so that the times between adjacent events from a given secondary process are independent and

[8] Photographs of Neyman and Bartlett can be found at the beginning of Chapter 10.

identically distributed. Both forms of cascaded processes are useful for generating fractal-rate point processes, as considered in Chapters 10 and 13.

Some cascaded point processes are isomorphic to doubly stochastic point processes, providing two different ways of viewing the same mathematical object (see, for example, Quenouille, 1949; Gurland, 1957; Bartlett, 1964; Lawrance, 1972; Neyman & Scott, 1972; Cox & Isham, 1980; Saleh & Teich, 1982; Teich & Diament, 1989; Lowen & Teich, 1991). The shot-noise-driven doubly stochastic Poisson point process, for example, is a particular Neyman–Scott cluster process, as discussed in Chapter 10.

4.6 BRANCHING POINT PROCESSES

Cascading need not be limited to two stages. Consider, for example, a shot-noise-driven doubly stochastic Poisson point process followed by a linear filter that converts the pulsatile sequence of events into a stochastic rate function suitable for driving a succeeding Poisson process, which is followed by another linear filter, and so on. The result is a cascade of shot-noise-driven doubly stochastic Poisson processes. The multifold statistics of the events at the output of an arbitrary number of such stages have been determined (Matsuo, Saleh & Teich, 1982). In accordance with expectations, the greater the number of stages, the larger the variability at the output.

The stages can also be constructed in such a way as to comprise **Thomas point processes** (Matsuo, Teich & Saleh, 1983), so that trigger events are carried forward from each stage to the next. In the limit as the number of such cascaded Thomas stages increases to infinity, while the mean number of added events per event of the previous stage becomes infinitesimal, the result converges to a **Poisson branching process** (Matsuo, Teich & Saleh, 1984). In particular, when the branching is instantaneous, the limit of continuous branching yields the **Yule–Furry branching process** with an initial Poisson population (Matsuo et al., 1984). The theory of branching processes, originally developed in connection with the survival of family names, has a long and august history in the annals of mathematics (Bienaymé, 1845; Watson & Galton, 1875; Yule, 1924; Furry, 1937; Kolmogorov & Dmitriev, 1947; Kendall, 1949, 1975; Harris, 1989).

4.7 LÉVY-DUST COUNTEREXAMPLE

We conclude this chapter with an example of a random collection of points that does *not* comprise an orderly point process. Mandelbrot (1982) coined the term **Lévy dust** to describe a particular random collection of points on a line segment. Its definition follows. Consider a finite-length segment of such a set, and count all the intervals between adjacent points that exceed a value ϵ; this number varies as ϵ^{-c} for some $c > 0$. All such intervals are independent of each other. Thus, the interval distribution exhibits scaling behavior and Lévy dusts belong to the class of one-dimensional fractal objects; indeed, they resemble randomized versions of the Cantor set.

Moreover, Lévy dusts resemble fractal renewal point processes (Chapter 7) in their interval independence and in their scaling behavior. However, these sets do not form orderly point processes. As ϵ becomes smaller, the number of intervals grows without limit; an infinite number of intervals lie in many segments. Consequently, all but an infinitesimal fraction of the intervals have a vanishing length. In particular, in any neighborhood about any point in the set, an infinite number of other points exist, violating the definition of an orderly point process.

Problems

4.1 *Normalized variance for an integrate-and-reset point process* Consider an integrate-and-reset point process with constant rate. All interevent intervals τ_k therefore assume the same fixed value τ, for all k, for this perfectly periodic point process. To render this point process stationary, while maintaining its periodic form, we randomize the absolute times. The time between the origin and the event that follows it is taken to be uniformly distributed in the interval $(0, \tau)$. Equivalently, the forward recurrence time from any fixed time s selected independently of the point process is given by

$$p_\tau(t) = \begin{cases} 1/\tau & 0 < t < \tau \\ 0 & \text{otherwise}. \end{cases} \tag{4.42}$$

4.1.1. Find an expression for the count variance, $\mathrm{Var}[Z(T)]$, and show that it cannot exceed $\frac{1}{4}$.

4.1.2. Find an expression for the normalized variance, $F(T)$.

4.2 *Time statistics of the homogeneous Poisson process* Let $dN(t)$ represent a homogeneous Poisson process with rate μ. Now choose a time v independently of $dN(t)$.

4.2.1. What is the probability density of the time remaining to the next event (forward recurrence time)?

4.2.2. What is the probability density of the time since the last event (the backward recurrence time)?

4.2.3. What is the probability density of the interval τ_* within which v lies?

4.2.4. Explain the difference between the expression derived immediately above and that in Eq. (4.3).

4.3 *Generalized dimensions for a homogeneous Poisson process* As indicated in Sec. 3.5.4, the generalized dimensions D_q assume integer values for nonfractal point processes. Consider the case of a homogeneous Poisson process $dN(t)$ with rate μ. Calculate expressions for $\mathrm{E}\big[\sum_k Z_k^q(T)\big]$ for a segment of that process of duration L. For both $q = 0$ and $q = 2$, find the associated limiting forms for large and small values of T. Verify, for both values of q, that $D_q = 0$ in the sense of Eq. (3.70), and that $D_q = 1$ in the sense of Eq. (3.72), thereby confirming that the homogeneous Poisson process is a nonfractal point process.

4.4 *Renewal process with exponential interval density* Use the steps listed below to demonstrate that a renewal point process constructed from exponentially distributed random variables, as in Eq. (4.3), satisfies Eqs. (4.1) and (4.2), and therefore must coincide with the homogeneous Poisson point process.

4.4.1. Show that the point-process spectrum follows the form of Eq. (4.9c) and use Eq. (3.59) to obtain the mean rate.

4.4.2. Show that Eq. (4.9d) provides the coincidence rate of this constructed process and that this establishes that nonoverlapping intervals are uncorrelated.

4.4.3. Extend this result to establish independence.

4.5 *Renewal-process spectrum at low frequencies* Prove Eq. (4.17).

4.6 *Count variance for renewal point processes* Show that substituting $k = 2$ in Eq. (4.19) indeed yields Eq. (4.21).

4.7 *Gamma renewal point process* A gamma probability density function takes the form (Parzen, 1962; Cox & Isham, 1980)

$$p_\tau(t) = \begin{cases} [\Gamma(\mathsf{m})]^{-1}\, \tau_0^{-\mathsf{m}}\, t^{\mathsf{m}-1}\, \exp(-t/\tau_0) & t > 0 \\ 0 & t \le 0, \end{cases} \tag{4.43}$$

where m is the *order* of the process[9] and $\Gamma(x)$ represents the (complete) Eulerian gamma function

$$\Gamma(x) \equiv \int_0^\infty t^{x-1}\, e^{-t}\, dt. \tag{4.44}$$

In general, the order of the gamma density can assume any positive real value, $0 < \mathsf{m} < \infty$.

4.7.1. Find the mean, variance, skewness, and kurtosis of the random variable associated with the probability density function in Eq. (4.43).

4.7.2. Suppose we construct a renewal point process using an interevent-interval probability density function specified by Eq. (4.43). Find the corresponding point-process spectrum.

4.7.3. For the particular case $\mathsf{m} = 2$, find the coincidence rate $G(t)$ as well as the count-based normalized variance $F(T)$ and normalized Haar-wavelet variance $A(T)$.

4.8 *Point-process and rate spectra for gamma renewal point processes* Equation (3.67) relates the rate spectrum $S_\lambda(f, T)$ to the point process spectrum $S_N(f)$. We examine this relation for two examples from the gamma-renewal-process family: $\mathsf{m} = 1$ (the homogeneous Poisson process), and $\mathsf{m} = 2$.

4.8.1. Calculate $S_\lambda(f, T)$ for the homogeneous Poisson process, and show that $S_\lambda(f, T)$ and $S_N(f)$ coincide in this particular case when $|f| < 1/T$.

[9] The integer-order gamma density is sometimes called the Erlang density, in honor of the Danish electrical engineer who used it to characterize waiting times associated with telephone calls (see Chapter 13 for a discussion of this issue and for a photograph of Erlang). This special form of the gamma density, along with its derivation, was known earlier (Ellis, 1844), but only in terms of a Gaussian approximation.

4.8.2. Repeat this exercise for the gamma renewal point process with $\mathsf{m} = 2$, and show that $S_\lambda(f, T)$ and $S_N(f)$ no longer agree.

4.8.3. Verify that the two measures do coincide, however, in the limit of small counting times T.

4.9 *Integrate-and-reset process with a gamma-distributed rate* An integrate-and-reset process is driven by a rate $\mu(t)$ that varies much more slowly than the time scale corresponding to the longest interevent interval.

4.9.1. Find an expression for the coefficient of variation C_τ for the interevent intervals, as defined in Eq. (3.5).

4.9.2. Evaluate C_τ for the special case of a gamma-distributed rate

$$p_\mu(x) = \begin{cases} [\Gamma(\mathsf{m})]^{-1}\,\mu_0^{-\mathsf{m}}\,x^{\mathsf{m}-1}\,\exp(-x/\mu_0) & x > 0 \\ 0 & x \le 0, \end{cases} \tag{4.45}$$

with $\mathsf{m} > 1$ and μ_0 a fixed, deterministic rate; the gamma function is defined in Prob. 4.7.

4.9.3. Why do we require $\mathsf{m} > 1$?

4.10 *Sinusoidally modulated point processes* Suppose we have a random rate defined by

$$\mu(t) = \mu_0\big[1 + \cos(\omega_0 t + \theta)\big], \tag{4.46}$$

with the random phase angle θ uniformly distributed in $(0, 2\pi]$ and μ_0 and ω_0 fixed, deterministic parameters. This renders the process ergodic and, in particular, stationary.

4.10.1. Let Eq. (4.46) be the rate of a Poisson point process. Find the mean value, coincidence rate, and count-based normalized variance for this doubly stochastic Poisson process.

4.10.2. Now let Eq. (4.46) serve as the rate for an integrate-and-reset point process, and assume that $\mu_0/\omega_0 \gg 1$. Ignoring values of μ_0 such that $2\pi\mu_0/\omega_0$ assumes a rational number, find an approximate expression for the interevent-interval probability density.

4.10.3. Attempt to calculate $\mathrm{E}[\tau^2]$ and explain which assumption breaks down in the process. Modify Eq. (4.46) to rectify the problem.

4.11 *Cascaded point process with Poisson primaries and secondaries* Let $dN_1(t)$ represent a homogeneous Poisson point process with rate μ_1. Suppose that every event k of $dN_1(t)$ triggers a secondary point process $dN_{2,k}(t)$, which has a duration τ_0 during which it generates events with a constant rate μ_2 as a segment of a homogeneous Poisson point process. Let all secondary points, taken together as indistinguishable events, form an output point process $dN_3(t)$. Assume that $\mu_1\tau_0 \ll 1$ so that we can safely ignore edge effects. By virtue of the memoryless property of the homogeneous Poisson process, this particular cascaded point process belongs both to the Neyman–Scott and Bartlett–Lewis families; choosing a random time or a random number of events is equivalent.

4.11.1. Find the mean, variance, and normalized variance of the number of points in an interval T such that $T/\tau_0 \gg 1$.

4.11.2. Imagine now that we modulate the primary process rate $\mu_1(t)$ to transmit information, setting $\mu_1(t) = \mu_1$ or $\mu_1(t) = 0$, and count the number of events $N_3(T)$. Here a $\mu_1(t) = \mu_1$ corresponds to a binary one, and $\mu_1(t) = 0$ corresponds to a binary zero. Find the probabilities of detecting a "one" when a "zero" was sent (an error known as a "false alarm"), and vice versa (an error known as a "miss").